

The local metric dimension of the lexicographic product of graphs

G. A. Barragán-Ramírez, A. Estrada-Moreno, Y. Ramírez-Cruz
and J. A. Rodríguez-Velázquez

Departament d'Enginyeria Informàtica i Matemàtiques,
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.

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Abstract

The metric dimension is quite a well-studied graph parameter. Recently, the adjacency dimension and the local metric dimension have been introduced and studied. In this paper, we give a general formula for the local metric dimension of the lexicographic product $G \circ \mathcal{H}$ of a connected graph G of order n and a family \mathcal{H} composed by n graphs. We show that the local metric dimension of $G \circ \mathcal{H}$ can be expressed in terms of the true twin equivalence classes of G and the local adjacency dimension of the graphs in \mathcal{H} .

Keywords: Local metric dimension; local adjacency dimension; lexicographic product graphs.

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1 Introduction

A metric generator of a metric space (X, d) is a set $S \subset X$ of points in the space with the property that every point of X is uniquely determined by the distances from the elements of S , *i.e.*, for every $x, y \in X$, there exists $z \in S$ such that $d(x, z) \neq d(y, z)$ [1]. In this case we say that z *distinguishes* the pair x, y .

Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \rightarrow \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between x and y and \mathbb{N} is the set of positive integers. Then (V, d_G) is a metric space since d_G satisfies (i) $d_G(x, x) = 0$ for all $x \in V$, (ii) $d_G(x, y) = d_G(y, x)$ for all $x, y \in V$ and (iii) $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ for all $x, y, z \in V$. A set $S \subset V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S . A minimum cardinality metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G , denoted by $\dim(G)$.

The notion of metric dimension of a graph was introduced by Slater in [25], where metric generators were called *locating sets*. Harary and Melter independently introduced the same concept in [13], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [18] and applications to chemistry in [16, 17]. Several variations of metric generators, including resolving dominating sets [2], independent resolving sets [3], local metric sets [20], strong resolving sets [24], adjacency resolving sets [15], k -metric/adjacency generators [5, 6], simultaneous (strong) metric generators [4, 22], etc., have since been introduced and studied.

A set S of vertices in a connected graph G is a *local metric generator* for G (also called local metric set for G [20]) if every two adjacent vertices of G are distinguished by some vertex of S . A minimum local metric generator is called a *local metric basis* for G and its cardinality, the *local metric dimension* of G , is denoted by $\dim_l(G)$.

The concept of adjacency generator¹ was introduced by Jannesari and Omoomi in [15] as a tool to study the metric dimension of lexicographic product graphs. A set $S \subset V$ of vertices in a graph $G = (V, E)$ is said to be an *adjacency generator* for G if for every two vertices $x, y \in V - S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y . A minimum cardinality adjacency generator is called an *adjacency basis* of G , and its cardinality the *adjacency dimension* of G , denoted by $\text{adim}(G)$ [15]. The concepts of *local adjacency generator*, *local adjacency basis* and *local adjacency dimension* are defined by analogy, and the local adjacency dimension of a graph G is denoted by $\text{adim}_l(G)$. This concept has been studied further by Fernau and Rodríguez-Velázquez in [9, 10] where they introduced the study of local adjacency generators and showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency dimension of H . As a consequence of this strong relation they showed that the problem of computing the local metric dimension and the (local) adjacency dimension of a graph is NP-hard.

As pointed out in [9, 10], any adjacency generator for a graph $G = (V, E)$ is also a metric generator in a suitably chosen metric space. Given a positive integer t , we define the distance function $d_{G,t} : V \times V \rightarrow \mathbb{N} \cup \{0\}$, where

$$d_{G,t}(x, y) = \min\{d_G(x, y), t\}. \quad (1)$$

Then any metric generator for $(V, d_{G,t})$ is a metric generator for $(V, d_{G,t+1})$ and, as a consequence, the metric dimension of $(V, d_{G,t+1})$ is less than or equal to the metric dimension of $(V, d_{G,t})$. In particular, the metric dimension of $(V, d_{G,1})$ equals $|V| - 1$, the metric dimension of $(V, d_{G,2})$ equals $\text{adim}(G)$ and, if G has diameter $D(G)$, then $d_{G,D(G)} = d_G$ and so the metric dimension of $(V, d_{G,D(G)})$ equals $\dim(G)$. Notice that when using the metric $d_{G,t}$ the concept of metric generator needs not be restricted to the case of connected graphs, as for any pair of vertices x, y belonging to different connected components of G we can assume that $d_G(x, y) = +\infty$ and so $d_{G,t}(x, y) = t$.

Notice that S is an adjacency generator for G if and only if S is an adjacency generator for its complement \overline{G} . This is justified by the fact that given an adjacency

¹Adjacency generators were called adjacency resolving sets in [15].

generator S for G , it holds that for every $x, y \in V - S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y , and this property holds in \overline{G} . Thus, $\text{adim}(G) = \text{adim}(\overline{G})$.

From the definitions of the different variants of generators, we can observe: an adjacency generator is a metric generator; a metric generator is a local metric generator; a local adjacency generator is a local metric generator; and an adjacency generator is a local adjacency generator. These facts show that the following inequalities hold for any graph G :

- (i) $\dim(G) \leq \text{adim}(G)$;
- (ii) $\dim_l(G) \leq \dim(G)$;
- (iii) $\dim_l(G) \leq \text{adim}_l(G)$;
- (iv) $\text{adim}_l(G) \leq \text{adim}(G)$.

Moreover, if $D(G) \leq 2$, then $\dim(G) = \text{adim}(G)$ and $\dim_l(G) = \text{adim}_l(G)$.

The radius of a graph G is denoted by $r(G)$. The following result describes situations with very small or large local adjacency dimensions.

Theorem 1. [9] *Let G be a non-empty graph of order t . The following assertions hold.*

- (i) $\text{adim}_l(G) = 1$ if and only if G is a bipartite graph having only one non-trivial connected component G^* and $r(G^*) \leq 2$.
- (ii) $\text{adim}_l(G) = t - 1$ if and only if $G \cong K_t$.

The remainder of the paper is structured as follows. After introducing some useful notation and terminology in Section 2, we extensively discuss our main results in Section 3. Then, Section 4 is devoted to show how all previous results presented in Section 3 in terms of the local adjacency dimension can be expressed in terms of the local metric dimension of graphs of the form $K_1 + H$. Finally, we show in Section 5 that the methodology used for studying the local metric dimension can be applied to the case of the local adjacency dimension of lexicographic product graphs. In particular, we discuss when the values of both dimensions coincide.

2 Preliminary concepts

Throughout the paper, we will use the notation K_n , $K_{r,n-r}$, C_n , N_n and P_n for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs of order n , respectively.

We use the notation $u \sim v$ if u and v are adjacent and $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G , $N_G(v)$ will denote the set of neighbours or *open neighbourhood* of v in G , i.e. $N_G(v) = \{u \in V(G) : u \sim v\}$.

The *closed neighbourhood* of v , denoted by $N_G[v]$, equals $N_G(v) \cup \{v\}$. If there is no ambiguity, we will simply write $N(v)$ or $N[v]$. Two vertices $x, y \in V(G)$ are *true twins* in G if $N_G[x] = N_G[y]$.

The subgraph of G induced by a set S of vertices is denoted by $\langle S \rangle_G$. If there is no ambiguity, we will simply write $\langle S \rangle$. The length of a shortest cycle (if any) in a graph G is called the girth of G , and it is denoted by $g(G)$. Acyclic graphs are considered to have infinite girth.

From now on we denote by \mathcal{G} the set of graphs H satisfying that for every local adjacency basis B , there exists $v \in V(H)$ such that $B \subseteq N_H(v)$. Notice that the only local adjacency basis of an empty graph N_r is the empty set, and so $N_r \in \mathcal{G}$. Moreover, $K_1 \cup K_2 \in \mathcal{G}$. In fact, a non-connected graph $H \in \mathcal{G}$ if and only if $H \cong N_r$ or $H \cong N_r \cup G$, where G is a connected graph in \mathcal{G} . We denote by Φ the family of empty graphs. Notice that $\Phi \subset \mathcal{G}$. On the other hand, it is readily seen that no graph of radius greater than or equal to four belongs to \mathcal{G} . As we will see in Proposition 16, if $H \in \mathcal{G}$ is a connected graph different from a tree, then $g(H) \leq 6$.

2.1 The lexicographic product $G \circ \mathcal{H}$

Let G be a graph of order n , and let $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ be an ordered family composed by n graphs. The *lexicographic product* of G and \mathcal{H} is the graph $G \circ \mathcal{H}$, such that $V(G \circ \mathcal{H}) = \bigcup_{u_i \in V(G)} (\{u_i\} \times V(H_i))$ and $(u_i, v_r)(u_j, v_s) \in E(G \circ \mathcal{H})$ if and only if $u_i u_j \in E(G)$ or $i = j$ and $v_r v_s \in E(H_i)$. Figure 1 shows the lexicographic product of P_3 and the family composed by $\{P_4, K_2, P_3\}$, and the lexicographic product of P_4 and the family $\{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$. In general, we can construct the graph $G \circ \mathcal{H}$ by taking one copy of each $H_i \in \mathcal{H}$ and joining by an edge every vertex of H_i with every vertex of H_j for every $u_i u_j \in E(G)$. Note that $G \circ \mathcal{H}$ is connected if and only if G is connected.

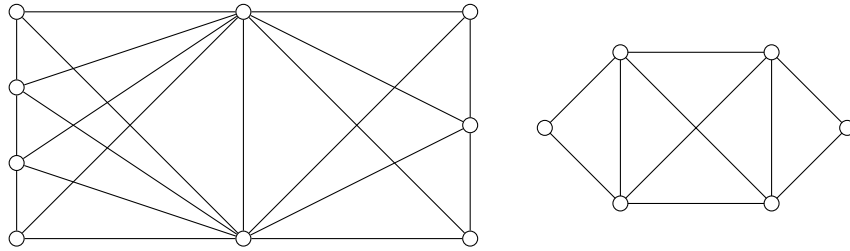


Figure 1: The lexicographic product graphs $P_3 \circ \{P_4, K_2, P_3\}$ and $P_4 \circ \{H_1, H_2, H_3, H_4\}$, where $H_1 \cong H_4 \cong K_1$ and $H_2 \cong H_3 \cong K_2$.

In particular, if $H_i \cong H$ for every $H_i \in \mathcal{H}$, then $G \circ \mathcal{H}$ is a standard lexicographic product graph, which is denoted as $G \circ H$ for simplicity. Another particular case of lexicographic product graphs is the join graph. The *join graph* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H [12, 26]. Note that $G + H \cong K_2 \circ \{G, H\}$. The join operation is commutative and associative. Now, for the sake of completeness, Figure 2 illustrates two examples of join graphs.

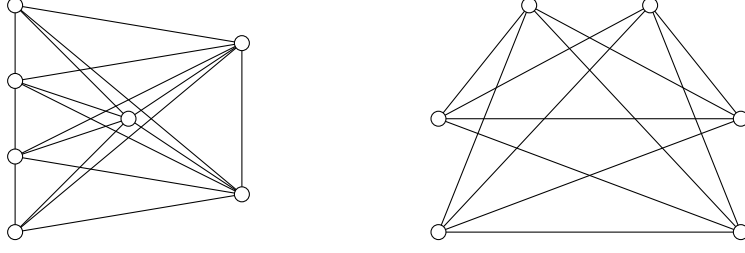


Figure 2: Two join graphs: $P_4 + C_3 \cong K_2 \circ \{P_4, C_3\}$ and $N_2 + N_2 + N_2 \cong K_3 \circ N_2$.

Moreover, complete k -partite graphs, $K_{p_1, p_2, \dots, p_k} \cong K_n \circ \{N_{p_1}, N_{p_2}, \dots, N_{p_k}\} \cong N_{p_1} + N_{p_2} + \dots + N_{p_k}$, are typical examples of join graphs. The particular case illustrated in Figure 2 (right hand side), is no other than the complete 3-partite graph $K_{2,2,2}$.

The relation between distances in a lexicographic product graph and those in its factors is presented in the following remark, for which it is necessary to recall (1).

Remark 2. *If G is a connected graph and (u_i, b) and (u_j, d) are vertices of $G \circ \mathcal{H}$, then*

$$d_{G \circ \mathcal{H}}((u_i, b), (u_j, d)) = \begin{cases} d_G(u_i, u_j), & \text{if } i \neq j, \\ d_{H_i, 2}(b, d), & \text{if } i = j. \end{cases}$$

We would point out that the remark above was stated in [11, 14] for the case where $H_i \cong H$ for all $H_i \in \mathcal{H}$.

The lexicographic product has been studied from different points of view in the literature. One of the most common researches focuses on finding relationships between the value of some invariant in the product and that of its factors. In this sense, we can find in the literature a large number of investigations on diverse topics. For instance, the metric dimension and related parameters have been studied in [7, 8, 15, 19, 21, 23]. For more information on product graphs we suggest the books [11, 14].

In order to state our main result (Theorem 3) we need to introduce some additional notation. Let $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ be the set of non-singleton true twin equivalence classes of a graph G . For the remainder of this paper we will assume that G is connected and has order $n \geq 2$, and $\mathcal{H} = \{H_1, \dots, H_n\}$. We now define the following sets and parameters:

- $T(G) = \bigcup_{j=1}^k U_j$.
- $V_E = \{u_i \in V(G) : H_i \in \Phi\}$.
- $I = \{u_i \in V(G) : H_i \in \mathcal{G}\}$.
- For any $I_j = I \cap U_j \neq \emptyset$, we can choose some $u \in I_j$ and set $I'_j = I_j - \{u\}$. We define the set $X_E = I - \bigcup_{I'_j \neq \emptyset} I'_j$.

- We say that two vertices $u_i, u_j \in X_E$ satisfy the relation \mathcal{R} if and only if $u_i \sim u_j$ and $d_G(u, u_i) = d_G(u, u_j)$ for all $u \in V(G) - (V_E \cup \{u_i, u_j\})$.
- We define \mathcal{A} as the family of sets $A \subseteq X_E$ such that for every pair of vertices $u_i, u_j \in X_E$ satisfying \mathcal{R} there exists a vertex in A that distinguishes them.
- $\varrho(G, \mathcal{H}) = \min_{A \in \mathcal{A}} \{|A|\}$.

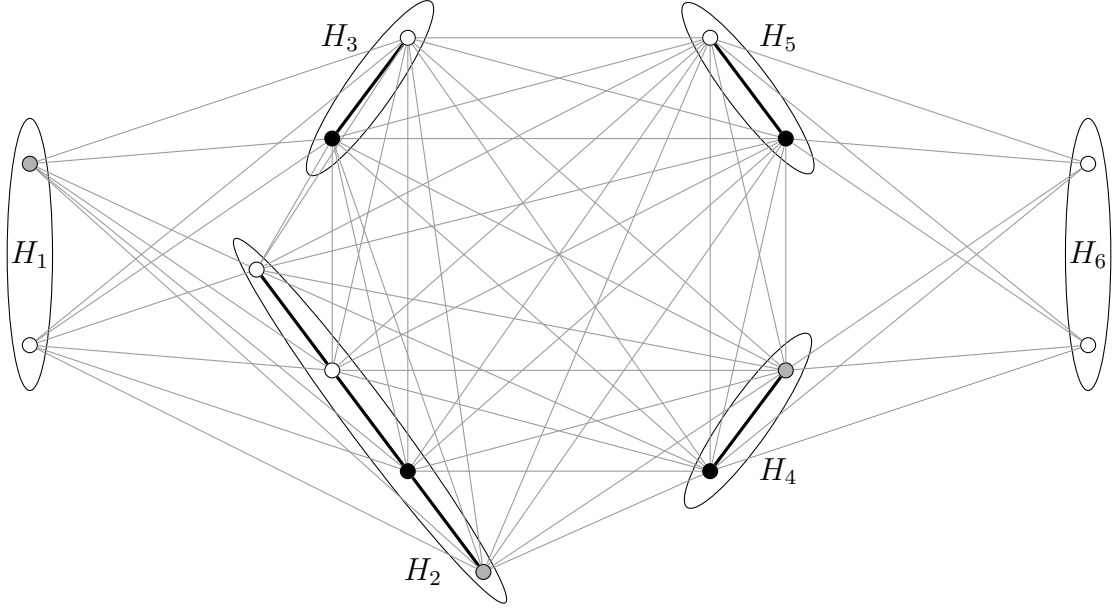


Figure 3: The graph $G \circ \mathcal{H}$, where G is the right-hand graph shown in Figure 1 and \mathcal{H} is the family composed by the graphs $H_1 \cong H_6 \cong N_2$, $H_2 \cong P_4$, $H_3 \cong H_4 \cong H_5 \cong K_2$. The set of black- and grey-coloured vertices is a local metric basis of $G \circ \mathcal{H}$.

With the aim of clarifying what this notation means, we proceed to show an example where we explain the role of these parameters when constructing a local metric generator W for a lexicographic product graph. Let G be the right-hand graph shown in Figure 1 and let \mathcal{H} be the family composed by the graphs $H_1 \cong H_6 \cong N_2$, $H_2 \cong P_4$, $H_3 \cong H_4 \cong H_5 \cong K_2$. Figure 3 shows the graph $G \circ \mathcal{H}$. Consider any $H_i \notin \Phi$. Note that the restriction of any local metric basis of $G \circ \mathcal{H}$ to the vertices of $\langle \{u_i\} \times V(H_i) \rangle \cong H_i$ must be a local adjacency generator for $\langle \{u_i\} \times V(H_i) \rangle$, as two adjacent vertices of $\langle \{u_i\} \times V(H_i) \rangle$ are not distinguished by any vertex outside $u_i \times V(H_i)$, so we can assume that the black-coloured vertices belong to W . Moreover, $U_1 = \{u_2, u_3\}$ and $U_2 = \{u_4, u_5\}$ are the non-singleton true twin equivalence classes of G . Since $u_4, u_5 \in I \cap U_2$, we have that no pair of non-black-coloured vertices in $(u_4 \times V(H_4)) \cup (u_5 \times V(H_5))$ is distinguished by any black-coloured vertex, so we add to W the grey-coloured vertex corresponding to the copy of H_4 and, by analogy, we add to W the grey-coloured vertex corresponding to the copy of H_2 . Besides, note that the white-coloured vertices of the copies of H_3 and H_5 are only distinguished by themselves and by vertices from the copies of H_1 and H_6 , so we need to add

one more vertex to W , e.g. the grey-coloured vertex in the copy of H_1 . Note that, according to our previous definitions, we have $V_E = \{u_1, u_6\}$ and we take $I'_1 = \{u_2\}$ and $I'_2 = \{u_4\}$. Thus, $X_E = \{u_1, u_3, u_5, u_6\}$. Therefore, since $u_1 \in X_E$ distinguishes the pair u_3, u_5 , the sole pair of vertices from X_E satisfying \mathcal{R} , we take $A = \{u_1\}$ and

conclude that $\varrho(G, \mathcal{H}) = 1$. Notice that, $\sum_{i=1}^6 \text{adim}_l(H_i) = 4$, $\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 2$

and $\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^6 \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}) = 7$.

3 Main results

Theorem 3. *Let G be a connected graph of order $n \geq 2$, let $\{U_1, U_2, \dots, U_k\}$ be the set of non-singleton true twin equivalence classes of G and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}).$$

Proof. We will first construct a local metric generator for $G \circ \mathcal{H}$. To this end, we need to introduce some notation. Let $V(G) = \{u_1, \dots, u_n\}$ and let S_i be a local adjacency basis of H_i , where $i \in \{1, \dots, n\}$. For any $I_j = I \cap U_j \neq \emptyset$, we choose $u \in I_j$ and set $I'_j = I_j - \{u\}$. Now, for every $u_i \in I'_j \neq \emptyset$, let $v_i \in V(H_i)$ such that $S_i \subseteq N_{H_i}(v_i)$. Finally, we consider a set $A \subseteq X_E$ achieving the minimum in the definition of $\varrho(G, \mathcal{H})$ and, for each $u_i \in A$, we choose one vertex $y_i \in V(H_i) - S_i$ such that $S_i \subseteq N_{H_i}(y_i)$. We claim that the set

$$S = \left(\bigcup_{S_i \neq \emptyset} (\{u_i\} \times S_i) \right) \cup \left(\bigcup_{I'_j \neq \emptyset} \{(u_i, v_i) : u_i \in I'_j\} \right) \cup \left(\bigcup_{u_i \in A} \{(u_i, y_i)\} \right)$$

is a local metric generator for $G \circ \mathcal{H}$. We differentiate the following four cases for two adjacent vertices $(u_i, v), (u_j, w) \in V(G \circ \mathcal{H}) - S$.

Case 1. $i = j$. In this case $v \sim w$. Since S_i is a local adjacency basis of H_i , there exists $x \in S_i$ such that $d_{H_i,2}(x, v) \neq d_{H_i,2}(x, w)$ and so for $(u_i, x) \in \{u_i\} \times S_i \subset S$ we have $d_{G \circ \mathcal{H}}((u_i, x), (u_i, v)) = d_{H_i,2}(x, v) \neq d_{H_i,2}(x, w) = d_{G \circ \mathcal{H}}((u_i, x), (u_i, w))$.

Case 2. $i \neq j$, $u_i, u_j \in U_l$ and $u_i \notin I_l$. For any $y \in S_i - N_{H_i}(v)$ we have that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G \circ \mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_i, y), (u_j, w))$.

Case 3. $i \neq j$, $u_i, u_j \in U_l$ and $u_i, u_j \in I_l$. If $v = v_i$ and $w = v_j$, then $(u_i, v_i) \in S$ or $(u_j, v_j) \in S$. If $v \neq v_i$ or $w \neq v_j$ (say $v \neq v_i$) then either $S_i \subseteq N_{H_i}(v)$, in which case $d_{G \circ \mathcal{H}}((u_i, v_i), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_i, v_i), (u_j, w))$, or there exists $y \in S_i - N_{H_i}(v)$ such that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G \circ \mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 =$

$$d_{G \circ \mathcal{H}}((u_i, y), (u_j, w)).$$

Case 4. $i \neq j$ and $N_G[u_i] \neq N_G[u_j]$. Notice that, in this case, $u_i \sim u_j$. If $u_i \notin I$, then $S_i \neq \emptyset$ and there exists $y \in S_i - N_{H_i}(v)$ such that $(u_i, y) \in \{u_i\} \times S_i \subseteq S$ and $d_{G \circ \mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_i, y), (u_j, w))$. Now, assume that $u_i, u_j \in I$. If $u_i \in I'_l$ or $u_j \in I'_l$ for some l (say $u_i \in I'_l$), then $d_{G \circ \mathcal{H}}((u_i, v_i), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_i, v_i), (u_j, w))$ or there exists $y \in S_i$ such that $d_{G \circ \mathcal{H}}((u_i, y), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_i, y), (u_j, w))$. Finally, if $u_i, u_j \notin \bigcup I'_l$, then by the construction of S there exists $u_l \in A \cup (V(G) - X_E)$ such that $d_G(u_l, u_i) \neq d_G(u_l, u_j)$. Since $u_l \in \{x : (x, y) \in S\}$, there exists $y \in V(H_l)$ such that $d_{G \circ \mathcal{H}}((u_l, y), (u_i, v)) \neq d_{G \circ \mathcal{H}}((u_l, y), (u_j, w))$.

In conclusion, S is a local metric generator for $G \circ \mathcal{H}$ and, as a result,

$$\dim_l(G \circ \mathcal{H}) \leq |S| = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I_j \neq \emptyset} (|I_j| - 1) + \varrho(G, \mathcal{H}).$$

It remains to show that $\dim_l(G \circ \mathcal{H}) \geq \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I_j \neq \emptyset} (|I_j| - 1) + \varrho(G, \mathcal{H})$. To

this end, we take a local metric basis W of $G \circ \mathcal{H}$ and for every $u_i \in V(G)$ we define the set $W_i = \{y : (u_i, y) \in W\}$. As for any $u_i \in V(G)$ and two adjacent vertices $v, w \in V(H_i)$, no vertex outside $\{u_i\} \times W_i$ distinguishes (u_i, v) and (u_i, w) , we can conclude that W_i is a local adjacency generator for H_i . Hence,

$$|W_i| \geq \text{adim}_l(H_i), \text{ for all } i \in \{1, \dots, n\}. \quad (2)$$

Now suppose, for the purpose of contradiction, that there exist $u_i, u_j \in I \cap U_l$ such that $|W_i| = \text{adim}_l(H_i)$ and $|W_j| = \text{adim}_l(H_j)$. In such a case, there exist $v_i \in V(H_i) - W_i$ and $v_j \in V(H_j) - W_j$ such that $W_i \subseteq N_{H_i}(v_i)$ and $W_j \subseteq N_{H_j}(v_j)$, which is a contradiction. Hence, if $|I \cap U_l| \geq 2$, then $|\{u_i \in I \cap U_l : |W_i| \geq \text{adim}_l(H_i) + 1\}| \geq |I \cap U_l| - 1$ and, as a consequence,

$$\sum_{u_i \in I \cap T(G)} |W_i| \geq \sum_{u_i \in I \cap T(G)} \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1). \quad (3)$$

On the other hand, assume that $\varrho(G, \mathcal{H}) \neq \emptyset$. We claim that

$$\sum_{u_j \in X_E} |W_j| \geq \sum_{u_j \in X_E} \text{adim}_l(H_j) + \varrho(G, \mathcal{H}). \quad (4)$$

To see this, we will prove that for any pair of vertices u_i, u_j satisfying \mathcal{R} there exists $u_r \in X_E$ such that $|W_r| \geq \text{adim}_l(H_r) + 1$. If $|W_i| = \text{adim}_l(H_i) + 1$ or $|W_j| = \text{adim}_l(H_j) + 1$, then we are done. Suppose that $|W_i| = \text{adim}_l(H_i)$ and $|W_j| = \text{adim}_l(H_j)$. Since W_i and W_j are local adjacency bases of H_i and H_j , respectively, there exist $v \in V(H_i)$ and $w \in V(H_j)$ such that $\{u_i\} \times W_i \subseteq N_{\langle \{u_i\} \times V(H_i) \rangle}(u_i, v)$ and $\{u_j\} \times W_j \subseteq N_{\langle \{u_j\} \times V(H_j) \rangle}(u_j, w)$. Thus, there exists $(u_r, y) \in \{u_r\} \times W_r$, $r \neq i, j$, which distinguishes the pair $(u_i, v), (u_j, w)$, and so $d_G(u_r, u_i) \neq d_G(u_r, u_j)$. Hence,

since u_i, u_j satisfy \mathcal{R} , we can claim that $u_r \in V_E \subseteq X_E$ and so $|W_r| > 0 = \text{adim}_l(H_r)$. In consequence, (4) holds. Therefore, (2), (3) and (4) lead to

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n |W_i| \geq \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho(G, \mathcal{H}),$$

as required. \square

From now on we proceed to obtain some particular cases of this main result. To begin with, we consider the case $\varrho(G, \mathcal{H}) = 0$.

Corollary 4. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. If for any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \mathcal{G}$ or $H_j \notin \mathcal{G}$, or there exists $u_l \in V(G)$ such that $H_l \notin \Phi$ and $d_G(u_l, u_i) \neq d_G(u_l, u_j)$, then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1).$$

In particular, if $\mathcal{H} \cap \Phi = \emptyset$, then $\varrho(G, \mathcal{H}) = 0$, and so we can state the following result, which is a particular case of Corollary 4.

Remark 5. *For any connected graph G of order $n \geq 2$ and any family $\mathcal{H} = \{H_1, \dots, H_n\}$ composed by non-empty graphs,*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1).$$

If $G \cong K_n$, then $\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = \max\{0, |I| - 1\}$, $|X_E| \in \{0, 1\}$, which implies that $\varrho(G, \mathcal{H}) = 0$, and so Theorem 3 leads to the following.

Corollary 6. *For any integer $n \geq 2$ and any family $\mathcal{H} = \{H_1, \dots, H_n\}$ of graphs,*

$$\dim_l(K_n \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \max\{0, |I| - 1\}.$$

Furthermore, the following assertions hold for a graph H .

- If $H \in \mathcal{G}$, then $\dim_l(K_n \circ H) = n \cdot \text{adim}_l(H) + n - 1$.
- If $H \notin \mathcal{G}$, then $\dim_l(K_n \circ H) = n \cdot \text{adim}_l(H)$.

Notice that, in the general case, $\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 0$ if and only if each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \mathcal{G}$. Thus, we can state the following corollary.

Corollary 7. Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. Then $\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i)$ if and only if for every two adjacent vertices $u_i, u_j \in I$, not belonging to the same true twin equivalence class, there exists $u \in V(G) - (V_E \cup \{u_i, u_j\})$ such that $d_G(u, u_i) \neq d_G(u, u_j)$ and each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \mathcal{G}$.

A particular case of the result above is stated in the next remark.

Remark 8. Let G be a connected bipartite graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. If $\mathcal{H} \not\subseteq \mathcal{G}$, then

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i).$$

Corollary 9. Let G be a connected bipartite graph of order n , let H be a non-empty graph, and let \mathcal{H} be a family composed by n graphs. If $\mathcal{H} - \Phi = \{H\}$, then

$$\dim_l(G \circ \mathcal{H}) = \begin{cases} \text{adim}_l(H) + 1, & \text{if } H \in \mathcal{G}; \\ \text{adim}_l(H), & \text{otherwise.} \end{cases}$$

Proof. If $G \cong K_2$, then $\varrho(G, \mathcal{H}) = 0$, $\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 1$ whenever $H \in \mathcal{G}$, and

$\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 0$ whenever $H \notin \mathcal{G}$. On the other hand, if $G \not\cong K_2$, then

$\sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) = 0$, $\varrho(G, \mathcal{H}) = 1$ whenever $H \in \mathcal{G}$, and $\varrho(G, \mathcal{H}) = 0$ whenever

$H \notin \mathcal{G}$. Since in any case $\sum_{i=1}^n \text{adim}_l(H_i) = \text{adim}_l(H)$, the result follows from Theorem 3. □

Our next result concerns the case of a family \mathcal{H} composed by empty graphs.

Remark 10. For any connected graph G of order $n \geq 2$ and any family \mathcal{H} composed by n graphs,

$$\dim_l(G \circ \mathcal{H}) \geq \dim_l(G).$$

In particular, if $\mathcal{H} \subset \Phi$, then

$$\dim_l(G \circ \mathcal{H}) = \dim_l(G).$$

Proof. Let W be a local metric basis of $G \circ \mathcal{H}$ and let $W_G = \{u : (u, v) \in W\}$ be the projection of W onto G . If there exist two adjacent vertices $u_i, u_j \in V(G) - W_G$ not distinguished by any vertex in W_G , then no pair of vertices $(u_i, v) \in \{u_i\} \times V(H_i)$, $(u_j, w) \in \{u_j\} \times V(H_j)$ is distinguished by elements of W , which is a contradiction. Thus, W_G is a local metric generator for G , so $\dim_l(G \circ \mathcal{H}) = |W| \geq |W_G| \geq \dim_l(G)$.

Now, we assume that $\mathcal{H} \subset \Phi$ and proceed to show that $\dim_l(G \circ \mathcal{H}) \leq \dim_l(G)$. Let A be a local metric basis of G . For each $H_l \in \mathcal{H}$ we select one vertex y_l and we define the set $A' = \{(u_l, y_l) : u_l \in A\}$. Let (u_i, v) and (u_j, w) be two adjacent vertices of $G \circ \mathcal{H}$. Since $u_i \sim u_j$, there exists $u_l \in A$ such that $d_G(u_i, u_l) \neq d_G(u_j, u_l)$. Now, if $l \neq i, j$, then we have $d_{G \circ \mathcal{H}}((u_l, y_l), (u_i, v)) = d_G(u_i, u_l) \neq d_G(u_j, u_l) = d_{G \circ \mathcal{H}}((u_l, y_l), (u_j, w))$. If $l = i$, then $d_{G \circ \mathcal{H}}((u_l, y_l), (u_i, v)) = 2 \neq 1 = d_{G \circ \mathcal{H}}((u_l, y_l), (u_j, w))$. Since the case $l = j$ is analogous to the previous one, we can conclude that A' is a local metric generator for $G \circ \mathcal{H}$ and, as a consequence, $\dim_l(G \circ \mathcal{H}) \leq \dim_l(G)$. Therefore, the proof is complete. \square

In general, the converse of Corollary 10 does not hold. For instance, we take G as the graph shown in Figure 4, $H_1 \cong H_5 \cong K_2$ and $H_2, H_3, H_4 \in \Phi$. In this case, we have that, for instance, $\{u_1, u_5\}$ is a local metric basis of G , whereas for any $y \in V(H_1)$ and $y' \in V(H_5)$, the set $\{(u_1, y), (u_5, y')\}$ is a local metric basis of $G \circ \mathcal{H}$, so $\dim_l(G \circ \mathcal{H}) = \dim_l(G) = 2$.

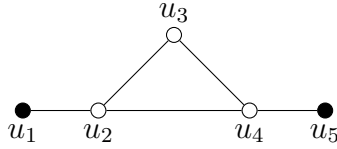


Figure 4: The set $\{u_1, u_5\}$ is a local metric basis of this graph.

As a direct consequence of Theorems 1 and 3 we deduce the following two results.

Theorem 11. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family composed by non-empty graphs. Then $\dim_l(G \circ \mathcal{H}) = n$ if and only if each true twin equivalence class of G contains at most one vertex u_i such that $H_i \in \mathcal{G}$ and each $H_i \in \mathcal{H}$ is a bipartite graph having only one non-trivial connected component H_i^* and $r(H_i^*) \leq 2$.*

Theorem 12. *Let G be a connected true twins free graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family composed by non-empty graphs of order n_i . Then $\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n n_i - n$ if and only if $H_i \cong K_{n_i}$, for all $H_i \in \mathcal{H}$.*

4 The local adjacency dimension of H versus the local metric dimension of $K_1 + H$

From now on we denote by \mathcal{G}' the set of graphs H satisfying that there exists a local metric basis of $K_1 + H$ which contains the vertex of K_1 .

Proposition 13. *Let H be a graph. Then $H \in \mathcal{G}'$ if and only if $H \in \mathcal{G}$.*

Proof. Let $H \in \mathcal{G}'$, and B a local metric basis of $\langle u \rangle + H$ such that $u \in B$. Since u does not distinguish any pair of vertices of H , $B - \{u\}$ is a local adjacency generator

for H , and so $\dim_l(\langle u \rangle + H) - 1 \geq \text{adim}_l(H)$. Now, if there exists a local adjacency basis A of H such that $A \not\subseteq N_H(v)$ for all $v \in V(H)$, then A is a local metric basis of $\langle u \rangle + H$ and so $\dim_l(\langle u \rangle + H) = \text{adim}_l(H)$, which is a contradiction. Therefore, $H \in \mathcal{G}$.

Now, let $H \in \mathcal{G}$. Suppose that there exists a local metric basis W of $\langle u \rangle + H$ such that $u \notin W$. In such a case, for every vertex $x \in V(H)$ there exists $y \in W$ such that $y \notin N_H(x)$, which implies that W is not a local adjacency basis of H , as $H \in \mathcal{G}$. Thus, since W is a local adjacency generator for H , we conclude that $\dim_l(\langle u \rangle + H) = |W| \geq \text{adim}_l(H) + 1$. Therefore, for any local adjacency basis A of H , $A \cup \{u\}$ is a local adjacency basis of $\langle u \rangle + H$. \square

Theorem 14. [9] *Let H be a non-empty graph. The following assertions hold.*

- (i) *If $H \notin \mathcal{G}'$, then $\text{adim}_l(H) = \dim_l(K_1 + H)$.*
- (ii) *If $H \in \mathcal{G}'$, then $\text{adim}_l(H) = \dim_l(K_1 + H) - 1$.*
- (iii) *If H has radius $r(H) \geq 4$, then $\text{adim}_l(H) = \dim_l(K_1 + H)$.*

As the following result shows, we can express all our previous results in terms of the local adjacency dimension of the graphs $K_1 + H_i$, where $H_i \in \mathcal{H}$, i.e., Theorem 15 is analogous to Theorem 3.

Theorem 15. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \dots, H_n\}$ a family of graphs. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \dim_l(K_1 + H_i) - \tau + \varrho(G, \mathcal{H}),$$

where τ is the number of non-singleton true twin equivalence classes of G having at least one vertex u_i such that $H_i \in \mathcal{G}'$.

Proof. Notice that, by Proposition 13, the parameter $\varrho(G, \mathcal{H})$ can be redefined in terms of \mathcal{G}' . The result immediately follows from Proposition 13 and Theorems 3 and 14. \square

Lemma 16. *Let H be a connected graph different from a tree. If $H \in \mathcal{G}$, then $g(H) \leq 6$.*

Proof. Let A be local adjacency basis of H . Since $H \in \mathcal{G}$, we consider v as the vertex of H such that $A \subseteq N_H(v)$. Let $N_i(v) = \{u \in V(H) : d_H(v, u) = i\}$. Since $A \subseteq N_1(v)$, we have that $N_3(v)$ is an independent set and $N_i(v) = \emptyset$, for all $i \geq 4$. Therefore, $g(H) \leq 6$. \square

By Proposition 13, Theorem 15 and Lemma 16 we can derive the following consequence of Theorem 15 (or equivalently, Theorem 3).

Corollary 17. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \dots, H_n\}$ a family composed by connected graphs. If each $H_i \in \mathcal{H}$ has radius $r(H_i) \geq 4$, or H_i is not a tree and it has girth $g(H_i) \geq 7$, then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^n \dim_l(K_1 + H_i) = \sum_{i=1}^n \text{adim}_l(H_i).$$

Proposition 18. [9] *For any integer $n \geq 4$, $\text{adim}_l(C_n) = \lceil \frac{n}{4} \rceil$.*

From Corollary 17 and Proposition 18 we deduce the following result.

Proposition 19. *Let G be a connected graph of order $t \geq 2$, and $\mathcal{H} = \{C_{n_1}, \dots, C_{n_t}\}$ a family composed by cycles of order at least 7. Then*

$$\dim_l(G \circ \mathcal{H}) = \sum_{i=1}^t \left\lceil \frac{n_i}{4} \right\rceil.$$

5 On the local adjacency dimension of lexicographic product graphs

By a simple transformation of Theorem 3 we obtain an analogous result on the local adjacency dimension of lexicographic product graphs, which we will state without proof. To this end, we consider again some of our previous notation. As above, let $\{U_1, U_2, \dots, U_k\}$ be the set of non-singleton true twin equivalence classes of a connected graph G of order $n \geq 2$, and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. Recall that $V_E = \{u_i \in V(G) - T(G) : H_i \in \Phi\}$, $I = \{u_i \in V(G) : H_i \in \mathcal{G}\}$ and, for any $I_j = I \cap U_j \neq \emptyset$, we can choose some $u \in I_j$ and set $I'_j = I_j - \{u\}$. Moreover, recall that $X_E = I - \bigcup_{I'_j \neq \emptyset} I'_j$. Now, we say that two vertices $u_i, u_j \in X_E$ satisfy the relation \mathcal{R}' if and only if $u_i \sim u_j$ and $d_{G,2}(u, u_i) = d_{G,2}(u, u_j)$ for all $u \in V(G) - (V_E \cup \{u_i, u_j\})$. We define \mathcal{A}' as the family of sets $A \subseteq X_E$ such that for every pair of vertices $u_i, u_j \in X_E$ satisfying \mathcal{R}' there exists a vertex in A that distinguishes them. Finally, we define $\varrho'(G, \mathcal{H}) = \min_{A \in \mathcal{A}'} \{|A|\}$.

Theorem 20. *Let G be a connected graph of order $n \geq 2$, let $\{U_1, U_2, \dots, U_k\}$ be the set of non-singleton true twin equivalence classes of G and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. Then*

$$\text{adim}_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1) + \varrho'(G, \mathcal{H}).$$

Let $G \cong P_4$ where $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $u_i \sim u_{i+1}$, for $i \in \{1, 2, 3\}$. If $H_1 \cong H_2 \cong H_4 \cong P_3$ and $H_3 \cong N_3$, then $\dim_l(G \circ \mathcal{H}) = 3 < 4 = \text{adim}_l(G \circ \mathcal{H})$. Notice that $\varrho(G, \mathcal{H}) = 0$ and $\varrho'(G, \mathcal{H}) = 1$. However, if $H_2 \cong H_3 \cong P_3$ and $H_1 \cong H_4 \cong N_3$, then $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H}) = 1$ and $\dim_l(G \circ \mathcal{H}) = 3 = \text{adim}_l(G \circ \mathcal{H})$.

We already know that for any graph G of diameter less than or equal to two, $\dim_l(G) = \text{adim}_l(G)$. However, the previous example shows that the above mentioned equality is not restrictive to graphs of diameter at most two, as $D(G \circ \mathcal{H}) = D(P_4) = 3$.

Notice that $\varrho'(G, \mathcal{H}) \geq \varrho(G, \mathcal{H})$, which is a direct consequence of Theorems 3 and 20, as well as the fact that $\text{adim}_l(G) \geq \dim_l(G)$ for any graph G . The next result corresponds to the case $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H})$.

Theorem 21. *Let G be a connected graph of order $n \geq 2$, and $\mathcal{H} = \{H_1, \dots, H_n\}$ a family of graphs. Then $\dim_l(G \circ \mathcal{H}) = \text{adim}_l(G \circ \mathcal{H})$ if and only if $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H})$.*

We now characterize the case $\varrho(G, \mathcal{H}) = \varrho'(G, \mathcal{H}) = 0$. The symmetric difference of two sets U and W will be denoted by $U \nabla W$.

Theorem 22. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family of graphs. Then the following assertions are equivalent.*

- (i) $\dim_l(G \circ \mathcal{H}) = \text{adim}_l(G \circ \mathcal{H}) = \sum_{i=1}^n \text{adim}_l(H_i) + \sum_{I \cap U_j \neq \emptyset} (|I \cap U_j| - 1)$.
- (ii) *For any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \mathcal{G}$ or $H_j \notin \mathcal{G}$, or there exists $u_l \in N_G(u_i) \nabla N_G(u_j)$ where H_l is not empty.*

Proof. By Theorems 3, 20 and 21, we only need to show that $\varrho'(G, \mathcal{H}) = 0$ if and only if (ii) holds.

((i) \Rightarrow (ii)) If $\varrho'(G, \mathcal{H}) = 0$, then for every two adjacent vertices $u_i, u_j \in I$, not belonging to the same true twin equivalence class, there exists $u_l \in V(G) - (V_E \cup \{u_i, u_j\})$ such that $d_{G,2}(u_l, u_i) \neq d_{G,2}(u_l, u_j)$, which implies that $u_l \in N_G(u_i) \nabla N_G(u_j)$ and H_l is not empty. Now, if $u_i, u_j \notin I$, then $H_i \notin \mathcal{G}$ or $H_j \notin \mathcal{G}$.

((ii) \Rightarrow (i)) If for any pair of adjacent vertices $u_i, u_j \in V(G)$, not belonging to the same true twin equivalence class, $H_i \notin \mathcal{G}$ or $H_j \notin \mathcal{G}$, or there exists $u_l \in N_G(u_i) \nabla N_G(u_j)$ where H_l is not empty, then no pair of adjacent vertices satisfy \mathcal{R}' and $V(G) - X_E$ is a local adjacency generator for G , which implies that $\varrho'(G, \mathcal{H}) = 0$. \square

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